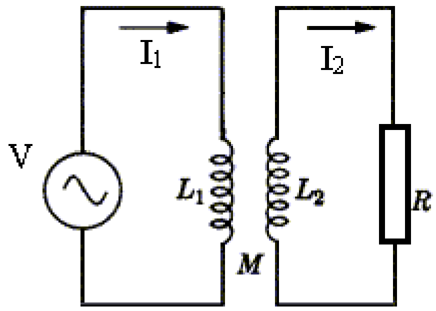
A mathematical study of two inductively coupled circuits shown below is presented here.



Let L_1 (L_2) be the inductance of the primary (secondary) coil and M be the mutual inductance between the two coils. The two coils are assumed to have zero resistance.

Loop the primary coil to get

$$L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V \qquad (1)$$

Similarly, for the secondary coil

$$RI_2 + L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = 0, \qquad (2)$$

If there is no flux leakage in the iron core,

$$L_1 = \alpha N_1^2,\ L_2 = \alpha N_2^2 \text{ and } M = \alpha N_1 N_2, \qquad (3)$$

where N_1 (N_2) is the number of turns of the primary (secondary) coil and α is a constant depending on the geometrical factors of the two coils.

Assume the A.C. source voltage varies with time in the form

$$V = V_o e^{i\omega t} \qquad (4)$$

With the complex amplitudes I_{1o} and I_{2o}, the two currents can be written as

$$I_1 = I_{1o} e^{i\omega t} \qquad (5i)$$

and

$$I_2 = I_{2o} e^{i\omega t} \qquad (5ii)$$

By using (5) and (6), (1) and (2) become

$$iX_1 I_{1o} + iX_M I_{2o} = V_0 \qquad (6)$$

$$RI_{2o} + iX_2 I_{2o} + iX_M I_{1o} = 0 \qquad (7)$$

where $X_1 = \omega L_1$, $X_2 = \omega L_2$ and $X_M = \omega M$.

Because of (3), $X_M^2 = X_1 X_2$ (8)

From the above equations, we obtain the following results.

A. Primary Current

Solve I_{1o} from (6) and (7), we get

$(A + iB)I_{1o} = V_o$ or

$$I_{1o} = \frac{V_o}{\sqrt{A^2 + B^2}} e^{i\phi},$$

where $A = \frac{X_M^2 R}{R^2 + X_2^2}$, $B = \frac{X_1 R^2}{R^2 + X_2^2}$ and $\phi = -\tan^{-1}(\frac{B}{A}) = -\tan^{-1}(\frac{R}{X_2})$ (9)

Note that $A^2 + B^2 = \frac{X_1^2 R^2}{R^2 + X_2^2}$.

Primary current $I_1 = \sqrt{\frac{R^2 + X_2^2}{R^2 X_1^2}} V_o e^{i(\omega t + \phi)}$ (10)

- When the **secondary circuit is open** (set R = ∞), the primary current is NOT zero but equal to

$$I_1 = \frac{V_o}{X_1} e^{i(\omega t - \frac{\pi}{2})}, \quad (11)$$

which is consistent with the result of regarding the primary coil as only a pure inductor ($V_0 = I_0 X_L$ and V leads I by $\pi/2$).

In this case, I_1 is small if X_1 is large.

- If the secondary circuit is closed with a load resistor R, the primary current increases immediately. A smaller R results a larger primary current I_1.
- When the load R is exceedingly large (open secondary coil), the primary current will lag behind the primary voltage by $\pi/2$. As R is reduced, the phase difference is decreased.
- As R tends to zero, the phase difference tends to zero too and I_1 tends to

$$I_1 = \sqrt{\frac{X_2^2}{R^2 X_1^2}} V_o e^{i\omega t} = \frac{X_2}{RX_1} V_o e^{i\omega t} = (\frac{N_2}{N_1})^2 \frac{V_o}{R} e^{i\omega t}$$

B. Voltage Ratio

Since $$V = L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} \qquad (12)$$

and $$RI_2 = -(L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt}) \qquad (13)$$

The p.d. across the primary coil (V_P) and that across the secondary coil (V_S) are now written in terms of N_1 and N_2 [by (3)],

$$V_P = \alpha N_1^2 \frac{dI_1}{dt} + \alpha N_1 N_2 \frac{dI_2}{dt} \qquad (14)$$

$$V_S = \alpha N_2^2 \frac{dI_2}{dt} + \alpha N_1 N_2 \frac{dI_1}{dt} \qquad (15)$$

Note: Only the magnitudes of the two voltages are considered, so the negative sign in (13), indicating the polarity of the coil, is dropped.

The turn ratio is obtained by dividing (14) by (15),

$$\frac{V_P}{V_S} = \frac{N_1}{N_2} \qquad (16)$$

The above result is true

- no matter the secondary coil is open or not
- no matter what a load resistor R is connected to the secondary coil.

C. Current Ratio:

From (7),

$$I_{2o} = -\frac{iX_M}{R + iX_2} I_{1o}$$

When the condition $R << X_2$ is satisfied, then

$$I_{2o} \approx -\frac{iX_M}{iX_2} I_{1o} = -\frac{X_M}{X_2} I_{1o}$$

Dropping the negative sign (we only concern their magnitudes, not their directions) and multiplying the factor $e^{i\omega t}$ back to I_{2o} and I_{1o}, we get

$$\frac{I_2}{I_1} = \frac{X_M}{X_2} = \frac{N_1}{N_2} \qquad (17)$$

The result $I_2/I_1 = N_1/N_2$ is true only when $R << X_2$ is satisfied.

Introductory physics textbooks usually derive this current ratio in this way:
Assume no energy loss, so input power = output power

$$V_1 I_1 = V_2 I_2$$

$$\frac{V_1}{V_2} = \frac{I_2}{I_1}$$

Because $\frac{V_1}{V_2} = \frac{N_1}{N_2}$, so $\frac{I_2}{I_1} = \frac{N_1}{N_2}$.

This derivation has implicitly assumed the primary current is in phase with the primary voltage. In general, they are not in phase [compare (4) and (10)], so the input power should be $V_1 I_1 \cos\phi$ (the factor $\cos\phi$ is called the power factor). Only when the condition $R << X_2$ is satisfied, the derivation is correct.